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BTZ Black Holes and Hawking Radiation

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Abstract

Hawking radiation emanating from a two-dimensional analogue of BTZ black holes is viewed as a tunnelling process. Two dimensional BTZ black holes (AdS(2) included) are treated as dynamical backgrounds in contrast to the standard methodology where the background geometry is fixed when evaluating Hawking radiation. This modification to the geometry gives rise to a nonthermal part in the radiation spectrum. Nonzero temperature of the extremal spinning BTZ black hole is found. The Bekenstein-Hawking area formula is easily derived for these dynamical geometries.

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Introduction

In 1992 Bañados, Teitelboim and Zanelli (BTZ) [1, 2] showed that (2+1)-dimensional gravity has a black hole solution. This black hole is described by two parameters, its mass M and its angular momentum (spin) J. It is locally anti-de-Sitter space and thus it differs from Schwarzschild and Kerr solutions in that it is asymptotically anti-de-Sitter instead of flat. Additionally it has no curvature singularity at the origin. AdS black holes, which are members of the two-parametric family of BTZ black holes, play a central role in AdS/CFT conjecture [3, 4] and also in brane-world scenarios [5, 6]. Specifically AdS(2) black hole is most interesting in the context of string theory and black hole physics [7, 8]. In this paper motivated by this recent interest in AdS(2) and/or in BTZ black hole backgrounds we treat the two-dimensional BTZ black holes (including AdS(2)) as radiating sources.
Concerning the quantum process called Hawking effect [9] much work has been done using a fixed background during the emission process. The idea of Keski-Vakkuri, Kraus and Wilczek (KKW) [10, 11, 12, 13] is to view the black hole background as dynamical by treating the Hawking radiation as a tunnelling process. The energy conservation is the key to this description. The total (ADM) mass is kept fixed while the mass of the BTZ black hole decreases due to the emitted radiation. The effect of this modification gives rise to additional terms in the formulae concerning the known results for the BTZ black holes; these additions are analogous to those found in [14, 15, 16, 17] for the respective geometries; a nonthermal partner ("greybody factor" [18, 19]) to the thermal spectrum of the Hawking radiation shows up. We explore the consequences to vacuum states of BTZ black holes (extremal BTZ black holes) since black holes are in general regarded as highly excited states. The extremality of the two-dimensional spinning ($J \neq 0$) BTZ black holes is now shifted since the angular momentum J of the BTZ black hole is been reached by the mass M earlier. This alteration produces a non -"frozen" extremal two-dimensional spinning BTZ black hole characterized by a constant temperature $T^{extremal}_{BTZ} \neq 0$. KKW method provides an easy way to derive the entropy of BTZ black holes.
In section 1 we make a short review of the two-dimensional spinning and spinless ($J = 0$) BTZ black holes and their properties. We present for the two-dimensional spinning BTZ black hole expressions for its temperature, area and entropy. The extremal two-dimensional spinless BTZ black hole is derived and its zero temperature is verified. AdS(2) black hole is produced as an isolated state (separated by a mass gap from the vacuum state - "massless" BTZ black hole, i.e. $M = 0$ and $J = 0$ - in the family of two-dimensional spinless BTZ black hole spectrum. In section 2 we implement the KKW method. Using

the imaginary part of the action of an outgoing positive-energy particle the temperature of the two-dimensional spinning BTZ black hole is evaluated and its dependence on the energy of the emitted massless particle is explicitly shown. This modified temperature due to the self-gravitation effect leads to a nonthermal spectrum. The extremal (vacuum state) two-dimensional spinning BTZ black hole is no more “frozen”. Its temperature is nonzero. Corresponding results for the spinless BTZ black holes are deduced but the corresponding extremal black hole has zero temperature since it is the state of zero mass. In section 3 we calculate the entropy of spinning and spinless BTZ black holes. Finally in section 4 we summarize our results and give our conclusions.

1 BTZ Black Holes

The black hole solutions of Bañados, Teitelboim and Zanelli in $(2+1)$ spacetime dimensions are derived from a three dimensional theory of gravity:

$$S = \int dx^3 \sqrt{-g}\,({}^{(3)}R + 2\Lambda) \tag{1}$$

with a negative cosmological constant $(\Lambda > 0)$. The corresponding line element is:

$$ds^2 = -\left(-M + \Lambda r^2 + \frac{J^2}{4r^2}\right)dt^2 + \frac{dr^2}{\left(-M + \Lambda r^2 + \frac{J^2}{4r^2}\right)} + r^2\left(d\theta - \frac{J}{2r^2}dt\right)^2 . \tag{2}$$

There are many ways to reduce the three dimensional BTZ black hole solutions to the two dimensional ones. The resulting two-dimensional line element is [20, 21]:

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 \tag{3}$$

where

$$g(r) = \left(-M + \Lambda r^2 + \frac{J^2}{4r^2}\right) \tag{4}$$

with M the ADM mass, J the angular momentum (spin) of the BTZ black holes and $-\infty < t < +\infty$, $0 \leq r < +\infty$.

For the positive mass black hole spectrum with spin $(J \neq 0)$, the line element (3) has two horizons:

$$r_{\pm}^2 = \frac{M \pm \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \tag{5}$$

with r_+, r_- the outer and inner horizon respectively. The area $\mathcal{A}_H$ and Hawking temperature T_H of the event (outer) horizon are [22, 23]:

$$\mathcal{A}_H = 2\pi\left(\frac{M + \sqrt{M^2 + \Lambda J^2}}{2\Lambda}\right)^{1/2}$$

$$= \quad 2\pi r_+ \tag{6}$$

$$T_H = \frac{\sqrt{2\Lambda}}{2\pi} \frac{\sqrt{M^2 - \Lambda J^2}}{(M + \sqrt{M^2 - \Lambda J^2})^{1/2}}$$
$$= \frac{\Lambda}{2\pi} \left(\frac{r_+^2 - r_-^2}{r_+} \right). \tag{7}$$

The entropy of the spinning BTZ black hole is:

$$S = 4\pi r_+ \tag{8}$$

and if we reinstate the Planck units (since in BTZ units $8\hbar G = 1$) we get:

$$S_{BH} = \frac{1}{4\hbar G} \mathcal{A}_H \tag{9}$$

which is the well-known Bekenstein-Hawking area formula for the entropy [25, 26, 27, 28] (or [24] by counting excited states).

Concerning the extremal spinning BTZ black hole:

$$M = \sqrt{\Lambda} J \tag{10}$$

the inner and outer horizon coincide ($r_+ = r_-$). This BTZ black hole can be viewed as the vacuum state of the positive mass spectrum of spinning BTZ black holes which saturates the bound:

$$M^2 - \Lambda J^2 \geq 0 \Leftrightarrow M \geq \sqrt{\Lambda} |J| \tag{11}$$

imposed in (5). Obviously the extremal spinning BTZ black hole has zero temperature ($T_H^{ext} = 0$).

For the spinless BTZ black hole the metric (3) becomes:

$$ds^2 = -\left(-M + \Lambda r^2\right) dt^2 + \left(-M + \Lambda r^2\right)^{-1} dr^2 \tag{12}$$

which has an horizon at:

$$r_H = \sqrt{\frac{M}{\Lambda}} \tag{13}$$

and is similar to Schwarzschild black hole with the important difference that it is not asymptotically flat but it has constant negative curvature. The temperature of the spinless BTZ black hole is:

$$T_H = \frac{\sqrt{2\Lambda}}{2\pi} M^{1/2}. \tag{14}$$

Spinning BTZ black holes with $M < \sqrt{\Lambda}|J|$ are discarded since they contain a naked singularity and for the same reason spinless BTZ black holes with $M < 0$ have not been treated above. The only exception is the spinless BTZ black hole with $M = -1$ and $J = 0$ which is the ordinary anti-de Sitter spacetime (AdS(2) black hole) separated by a mass gap from the "massless" BTZ black hole ($M = 0$ and $J = 0$).

2 KKW Methodology

In order to apply the idea of Keski-Vakkuri, Kraus and Wilczek (KKW) [10, 11, 12, 13] to the positive mass BTZ black hole spectrum with fixed J (3-4) we have to make a coordinate transformation. We choose the Painlevé coordinates [29] (utilized for black hole backgrounds recently in [30]) which are non-singular on the outer horizon (r_+). Thus we will be able to deal with phenomena whose main contributions come from the outer horizon. We introduce the time coordinate τ_P by imposing the ansätz:

$$\sqrt{g(r)}\,dt = \sqrt{g(r)}\,d\tau_P - \sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}}\,\frac{dr^2}{\sqrt{g(r)}}. \tag{15}$$

The line element (3-4) is now written as :

$$ds^2 = -\left(-M + \Lambda r^2 + \frac{J^2}{4r^2}\right) d\tau_P^2 + 2\sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}}\,d\tau_P dr + dr^2. \tag{16}$$

It is obvious from the above expression that there is no singularity at the points r_+ and r_-. The radial null ($ds^2 = 0$) geodesics followed by a massless particle are:

$$\dot{r} \equiv \frac{dr}{d\tau_P} = \pm 1 - \sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}} \tag{17}$$

where the signs $+$ and $-$ correspond to the outgoing and ingoing geodesics, respectively, under the assumption that τ_P increases towards future.
We fix the total ADM mass and we let the mass M of the BTZ black hole vary. If a shell of energy (mass) ω is radiated outwards the outer horizon then the BTZ black hole mass will be reduced to $M - \omega$ and the shell of energy will travel on the modified geodesics:

$$\dot{r} = 1 - \sqrt{1 + (M-\omega) - \Lambda r^2 - \frac{J^2}{4r^2}} \tag{18}$$

produced by the modified line element:

$$ds^2 = -\left(-(M-\omega) + \Lambda r^2 + \frac{J^2}{4r^2}\right) d\tau_P^2 + 2\sqrt{1 + (M-\omega) - \Lambda r^2 - \frac{J^2}{4r^2}}\,d\tau_P dr + dr^2. \tag{19}$$

It is known that the emission rate from a radiating source can be expressed in terms of the imaginary part of the action for an outgoing positive-energy particle as :

$$\Gamma = e^{-2ImS} \tag{20}$$

but also in terms of the temperature and the entropy of the radiating source which in our case will be a BTZ black hole :

$$\Gamma = e^{-\beta\omega} = e^{+\Delta S_{BTZ}} \tag{21}$$

where β is the inverse temperature of the BTZ black hole and ΔS_{BTZ} is the change the entropy of the BTZ black hole before and after the emission of the shell of energy ω (outgoing massless particle). It is clear that if we evaluate the action then we will know the temperature and/or the change in the entropy of the BTZ black hole. We therefore evaluate the imaginary part of the action for an outgoing positive-energy particle which crosses the event horizon outwards from :

$$r_{in} = r_+(M, J) = \left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right) \tag{22}$$

to

$$r_{out} = r_+(M - \omega, J) = \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right). \tag{23}$$

The imaginary part of the action is:

$$ImS = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp'_r dr. \tag{24}$$

We make the transition from the momentum variable to the energy variable using Hamilton's equation $\dot{r} = \frac{dH}{dp_r}$ and equation (18). The result is :

$$ImS = Im \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{(-d\omega')dr}{1 - \sqrt{1 + (M - \omega') - \Lambda r^2 - \frac{J^2}{4r^2}}} \tag{25}$$

where the minus sign is due to the Hamiltonian being equal to the modified mass $H = M - \omega$. This is not disturbing since $r_{in} > r_{out}$. After some calculations (involving contour integration into the lower half of ω' plane) we get :

$$ImS = 2\pi \left[\left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} - \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \right]. \tag{26}$$

Apparently the emission rate depends not only on the mass M and angular momentum (spin) J of the BTZ black hole but also on the energy ω of the emitted massless particle :

$$\begin{aligned} \Gamma(\omega, M, J) &= e^{-2ImS} \\ &= exp \left[4\pi \left(\sqrt{\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda}} - \sqrt{\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda}} \right) \right]. \end{aligned} \tag{27}$$

Comparing (21) and (27) we deduce that the modified temperature (due to the self-gravitation) of the BTZ black hole is :

$$T(\omega) = \frac{\omega}{4\pi} \left[\left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} - \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \right]^{-1}. \tag{28}$$

We see that there are modifications to the result previously mentioned (reftemp1) for a fixed BTZ black hole background. The temperature of the BTZ black hole is no longer the Hawking temperature T_H. These modifications comprise the so-called "greybody-factor" that measures the departure from the pure blackbody spectrum and lead to a nonthermal spectrum [18, 19].
If we evaluate the temperature to first order in ω we get:

2.1 Spinning ($J \neq 0$) BTZ black hole

It is a welcomed and not unexpected feature the fact that we reach the known expression (7) for the temperature:

$$T_H = \frac{\sqrt{2\Lambda}}{2\pi} \frac{\sqrt{M^2 - \Lambda J^2}}{(M + \sqrt{M^2 - \Lambda J^2})^{1/2}} \ . \tag{29}$$

This constitutes another evidence for our arguments placed here.
Concerning the extremal (vacuum solution) spinning BTZ black hole the condition for extremality now satisfied will be:

$$M - \omega = \sqrt{\Lambda} J \ . \tag{30}$$

This modification indicates that the mass M of the spinning BTZ black hole cannot be less than the angular momentum (spin) J (since $\omega = M - \sqrt{\Lambda} J > 0$) and the temperature of the extremal spinning BTZ black hole will not be zero but:

$$T_{BTZ}^{extremal} = \frac{\sqrt{2\Lambda}}{4\pi} \frac{(M - \sqrt{\Lambda} J)}{(M + \sqrt{M^2 - \Lambda J})^{1/2} - (\sqrt{\Lambda} J)^{1/2}} \ . \tag{31}$$

2.2 Spinless ($J = 0$) BTZ black hole

The modified temperature (28) for the spinless BTZ black hole becomes:

$$T(\omega) = \frac{\omega}{4\pi} \left[\sqrt{\frac{M}{\Lambda}} - \sqrt{\frac{M - \omega}{\Lambda}} \right] \tag{32}$$

and which in first order in ω is:

$$T_H = \frac{\sqrt{\Lambda}}{2\pi} \, M^{1/2} \tag{33}$$

in agreement to what was shown in [15].
Concerning the extremal (vacuum solution) spinless BTZ black hole, the condition for extremality satisfied is $M = 0$ and the corresponding temperature of the vacuum state for the $J = 0$ family is also zero:

$$T_H = 0 \ . \tag{34}$$

In this case the line element (3-4) takes the form:

$$ds^2 = -\Lambda dt^2 + \frac{dr^2}{\Lambda r^2} \tag{35}$$

describing a “naked singularity”.
For the case $M = -1$ which may be recognized as the ordinary anti-de-Sitter space (AdS(2) spacetime) the modified temperature is:

$$T(\omega) = \frac{\omega\sqrt{\Lambda}}{4\pi}\left(1 - \sqrt{1-\omega}\right)^{-1} \tag{36}$$

which, to first order in ω, gives:

$$T_H = \frac{\sqrt{\Lambda}}{2\pi} \ . \tag{37}$$

3 Entropy Calculation via KKW Method

It is obvious that we can have a short and direct derivation of the entropy of BTZ black hole up to a constant using equations (20) and (21) where :

$$\Delta S_{BTZ} = S(M-\omega) - S(M). \tag{38}$$

Indeed, if we combine (20), (21) and (28) we get the known expression (8) up to a constant for the entropy of the spinning BTZ black hole:

$$S(M) = 4\pi\left[\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda}\right]^{1/2} + S_0 \tag{39}$$

where S_0 is the arbitrary constant.
For the spinless BTZ black hole the entropy is:

$$S(M) = 4\pi\sqrt{\frac{M}{\Lambda}} + S_0' \tag{40}$$

where S_0' is also an arbitrary constant. If we adopt the conjecture that the entropy of an excited state tends to the entropy of its vacuum state [31] then:

$$S_0 = S_{BTZ}^{extremal} \tag{41}$$

since the temperature of the spinning BTZ black hole is no more zero and for the spinless BTZ black holes:

$$S_0' = 0 \tag{42}$$

since its vacuum state is the state of zero mass.

Discussion

In this work, we have viewed the Hawking radiation as a quantum tunnelling process. The self-gravitation of the radiation was included and this treatment introduced a nonthermal part for the radiation spectrum of the BTZ black holes. The temperature of the two-dimensional spinning BTZ black hole is no more the Hawking temperature and the "greybody factor" showing up declares explicitly the dependence of the temperature on the emitted particle's energy ω. The leading term in ω gives the thermal Boltzmann factor while the higher order terms represent corrections emanating from the response of the background geometry to the emission of a quantum. The extremal two-dimensional spinning BTZ black hole is no more "frozen" but it carries a background temperature $T_{BTZ}^{extremal}$ ensuring the validity of the third law of black hole thermodynamics [32]. Therefore it is obvious that we again have a strong evidence to believe that black holes constitute excited states while the extremal black holes correspond to ground (vacuum) states.

The above-mentioned treatment for incorporating the effects of the emission of a shell of energy for the case of two-dimensional spinless BTZ black holes (including AdS(2) black hole) yields the corresponding modified temperature with the leading term in ω again being the thermal Boltzmann factor. It is interesting that for the case of the extremal two-dimensional spinless BTZ black hole the above procedure yields precisely the result obtained with a fixed background, i.e. zero temperature. The reason for this lack of "greybody factor" in this case is the non - dependence of the specific background ("massless" BTZ black hole) on black hole mass .

Additionally the imaginary part of the action of the outgoing positive-energy particle is linear in the change of the entropy. We derive in a short and direct way the entropy for the spinning and spinless BTZ black hole up to a constant which is the entropy of the corresponding vacuum states. The results obtained are in agreement with those for a fixed BTZ black hole background. We believe that the reason of this coincidence namely the lack of dependence of the entropy on the emitted shell of energy can be traced to the following facts: (i) the KKW method does not modify the action by addition of terms but rather by fine-tuning the action integral's limits and (ii) the θ coordinate has been suppressed.

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